A comparison of Several Models of Weighted Tree Automata

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Terms (= trees)

Ranked alphabet : $(\Sigma, rank)$ with $rank : \Sigma \to \mathbb{N}$

$\Sigma^{(m)} = \{\sigma \in \Sigma \mid rank(\sigma) = m\}$

The set of terms (trees) over Σ and a set A is the smallest set $T_\Sigma(A)$ satisfying:

(i) $\Sigma^{(0)} \cup A \subseteq T_\Sigma(A)$,

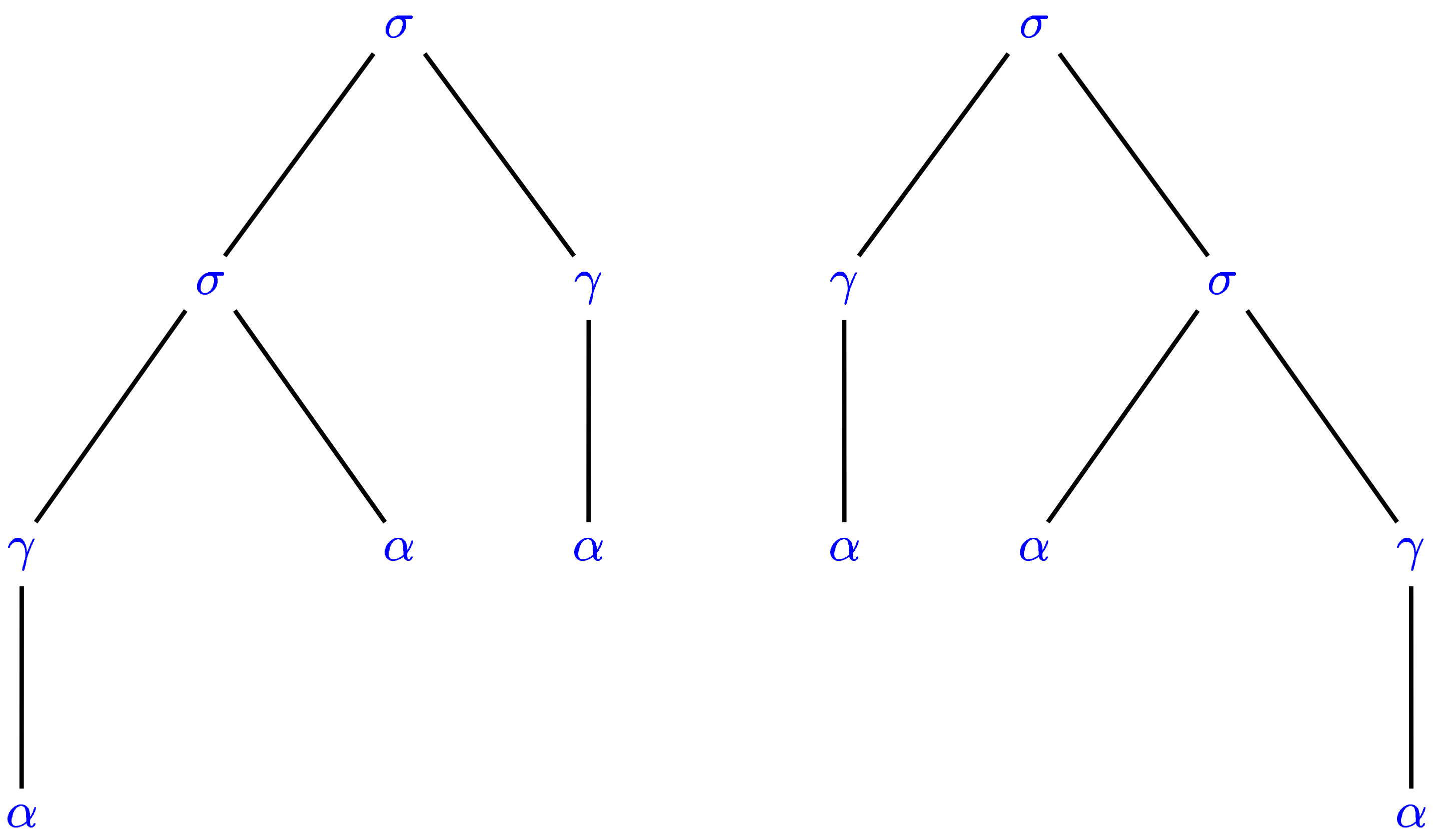
(ii) if $k \geq 1$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_m \in T_\Sigma(A)$, then $\sigma(t_1, \ldots, t_m) \in T_\Sigma(A)$.

$T_\Sigma = T_\Sigma(\emptyset)$ We have $T_\Sigma \neq \emptyset$ iff $\Sigma^{(0)} \neq \emptyset$.

Tree language : $L \subseteq T_\Sigma$ (or: $L : T_\Sigma \to \{0,1\}$) .

Trees (= terms)

Example: $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$



Nondeterministic Tree Automata

The classical definition.

A *(finite-state bottom-up) tree automaton* is a tuple $M = (Q, \Sigma, F, \delta)$, where

- Q is a finite set (*states*),
- Σ is a ranked alphabet (*input ranked alphabet*),
- $F \subseteq Q$ is a set (*final states*), and
- δ is a family $(\delta_\sigma | \sigma \in \Sigma)$ of mappings $\delta_\sigma : Q^m \to \mathcal{P}(Q)$ for $\sigma \in \Sigma^{(m)}$.

M is deterministic if $|\delta_\sigma(q_1, \ldots, q_m)|$ has at most one element for all $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $q_1, \ldots, q_m \in Q$.

The family δ extends to a mapping $\delta_M : T_\Sigma \to \mathcal{P}(Q)$. The *tree language recognized by* M is

$$L_M = \{s \in T_\Sigma \mid \delta_M(s) \cap F \neq \emptyset\}.$$

Nondeterministic Tree Automata

Examples of recognizable tree languages:

- the set of derivation trees of a cf grammar
- set of trees which contain the pattern $\sigma(\bullet, \alpha)$
- many other examples

Theorem. Bottom-up tree automata and deterministic bottom-up tree automata have the same recognizing power.

Proof. The standard powerset construction.

Nondeterministic Tree Automata

Example.

$\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$, show that $L = \{s \in T_\Sigma \mid \sigma(\bullet, \alpha) \text{ occurs in } s\}$ is recognizable.

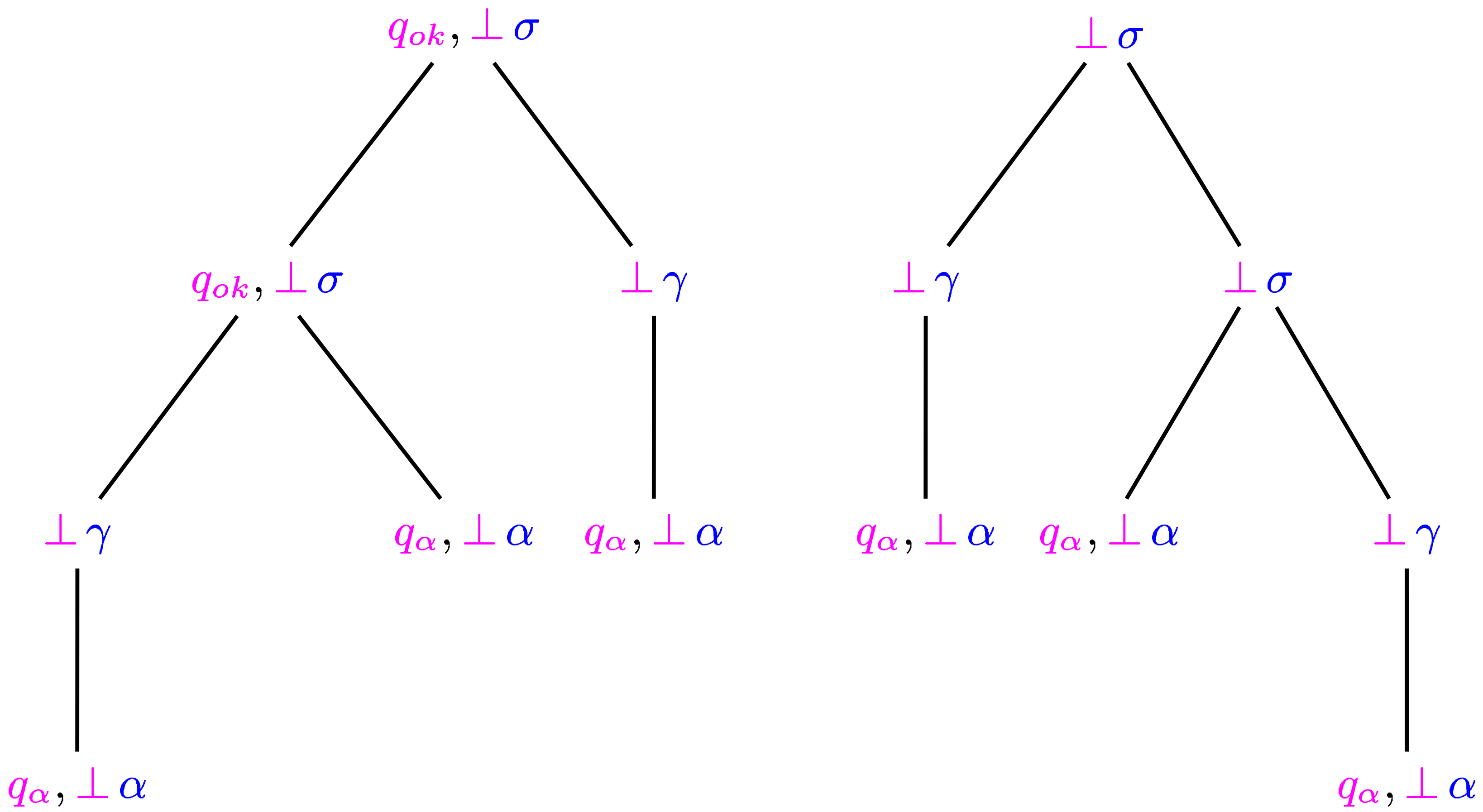
Let $M = (Q, \Sigma, F, \delta)$, where

- $Q = \{\perp, q_\alpha, q_{ok}\}$,
- $F = \{q_{ok}\}$,
- - $\delta_\alpha = \{\perp, q_\alpha\}$,
  - $\delta_\sigma(\perp, q_\alpha) = \delta_\sigma(-, q_{ok}) = \delta_\sigma(q_{ok}, -) = \{q_{ok}\}$, otherwise $\delta_\sigma(-, -) = \{\perp\}$,
  - $\delta_\gamma(q_{ok}) = \{q_{ok}\}$, otherwise $\delta_\gamma(-) = \{\perp\}$.

Then $L_M = L$.

Nondeterministic Tree Automata

Example.



Semirings

Semiring : $(K, \oplus, \odot, 0, 1)$

- $(K, \oplus, 0)$ is a commutative monoid,
- $(K, \odot, 1)$ is a monoid,

  and for every $a, b, c \in K$:
$$(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$$
$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$$
$$a \odot 0 = 0 \odot a = 0.$$

Examples :

- Boolean semiring : $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$
- semiring of natural numbers : $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$
- semiring of formal languages : (over Δ) $\mathrm{Lang}_\Delta = (\mathcal{P}(\Delta^*), \cup, \cdot, \emptyset, \{\varepsilon\})$
- tropical semiring : $\mathrm{Trop} = (\mathbb{N} \cup \{\infty\}, min, +, \infty, 0)$
- arctic semiring : $\mathrm{Arct} = (\mathbb{N} \cup \{-\infty\}, max, +, -\infty, 0)$

Nondeterministic Tree Automata

An algebraic definition

A system $M = (Q, \Sigma, F, \delta)$ (over $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in \{0,1\}$

$\delta = (\delta_m : \Sigma^{(m)} \to \{0,1\}^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$
\begin{array}{ccc} & & \begin{array}{ccc} \dots & q & \dots \end{array} \\
 & \vdots & \\
\delta_m(\sigma) & = \;\; q_1 \dots q_m & \left[\begin{array}{ccc} & \vdots & \\ \dots & 0/1 & \dots \\ & \vdots & \end{array} \right] \in \{0,1\}^{Q^m \times Q} \\
 & \vdots &
\end{array}
$$

Note, equivalent with $\delta_\sigma : Q^m \to \mathcal{P}(Q)$.

M is deterministic if, for every $q_1, \dots, q_m \in Q$, there is at most one q with $\delta_m(\sigma)_{q_1 \dots q_m, q} \neq 0$.

Nondeterministic Tree Automata

An algebraic definition

We associate the Σ-algebra $\mathcal{A}_M = (\{0,1\}^Q, \overline{\Sigma}_\delta)$, where $\overline{\Sigma}_\delta = (\overline{\delta_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$ and

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigvee_{q_1,\ldots,q_m \in Q} (v_1)_{q_1} \wedge \ldots \wedge (v_m)_{q_m} \wedge \delta_m(\sigma)_{q_1 \ldots q_m, q}.$$

Let $h_\delta : T_\Sigma \to \{0,1\}^Q$ be the *unique* Σ*-homomorphism from* T_Σ *to* $\mathcal{A}_M$.

The tree language recognized by M is $L_M : T_\Sigma \to \{0,1\}$ defined, for every $s \in T_\Sigma$, by

$$L_M(s) = \bigvee_{q \in Q} h_\delta(s)_q \wedge F_q.$$

Tree series

(Tree language : $L : T_\Sigma \to \{0, 1\}$)

Tree series : $\varphi : T_\Sigma \to K$, where $(K, \oplus, \odot, 0, 1)$ is a semiring

Examples of tree series:

$\mathrm{height} : T_\Sigma \to \mathbb{N}$, in $\underline{\mathrm{Arct}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

$\mathrm{size}_\sigma : T_\Sigma \to \mathbb{N}$, in $\underline{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$

$\mathrm{size} : T_\Sigma \to \mathbb{N}$, in $\underline{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$

$\#_{\sigma(\bullet,\alpha)} : T_\Sigma \to \mathbb{N}$, in $\underline{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$

$\mathrm{shortest}_\alpha : T_\Sigma \to \mathbb{N}$, in $\underline{\mathrm{Trop}} = (\mathbb{N} \cup \{-\infty\}, \min, +, -\infty, 0)$

$\mathrm{yield} : T_\Sigma \to \mathcal{P}(\Sigma^*)$, in $\mathrm{Lang}_\Sigma = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$

$\mathrm{pos} : T_\Sigma \to \mathcal{P}(\mathbb{N}^*)$, in $\mathrm{Lang}_\mathbb{N}$

$\mathrm{pos}_{\sigma(\bullet,\alpha)} : T_\Sigma \to \mathcal{P}(\mathbb{N}^*)$, in $\mathrm{Lang}_\mathbb{N}$

Tree series

The set of tree series over K and Σ is denoted by $K\langle\langle T_\Sigma \rangle\rangle$.

For $s \in T_\Sigma$, we write (φ, s) for $\varphi(s)$.

The *support* of φ is $\mathrm{supp}(\varphi) = \{s \in T_\Sigma \mid (\varphi, s) \neq 0\}$.

The tree series φ is *polynomial* if $\mathrm{supp}(\varphi)$ is finite.

The set of polynomial tree series over K and Σ is denoted by $K\langle T_\Sigma \rangle$.

Generalizations

- recognizability by multilinear mappings over some finite dimensional K-vector space, where K is a field, cf. [BR82],
- recognizability by K-Σ-tree automata, where K is a commutative semiring, cf. [Boz99],
- recognizability by weighted tree automata over K, where K is a semiring, cf. [AB87],
- recognizability by finite tree automata over K with fixpoint semantics, where K is a commutative and continuous semiring, cf. [Kui98, ÉK03],
- recognizability by polynomially-weighted tree automata, where K is a semiring, cf. [Sei92, Sei94], and
- recognizability by weighted tree automata over M-monoids, cf. [Mal05] and [FMV06].

K-semimodule:

$(K, \oplus, \odot, 0, 1)$ a commutative semiring, $(V, +, 0)$ a commutative monoid, and $\cdot : K \times V \to V$ a function:

$$(k \odot k') \cdot v = k \cdot (k' \cdot v) \tag{1}$$

$$k \cdot (v + v') = (k \cdot v) + (k \cdot v') \tag{2}$$

$$(k \oplus k') \cdot v = (k \cdot v) + (k' \cdot v) \tag{3}$$

$$1 \cdot v = v \tag{4}$$

$$k \cdot 0 = 0 \cdot v = 0 \tag{5}$$

K-vector space: K is a field and V is a group

A mapping $\omega : V^m \to V$ is *multilinear* if:

$\omega(v_1, \ldots, v_{i-1}, kv + k'v', v_{i+1}, \ldots, v_m) = k\omega(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_m) + k'\omega(v_1, \ldots, v_{i-1}, v', v_{i+1}, \ldots, v_m)$

Multilinear mappings over finite-dimensional vector spaces

A *multilinear representation* [BR82] of T_Σ is (V, μ, γ) where

- $(V, +, 0)$ is a finite-dimensional K-vector space (K is a field),
- transitions: $\mu = (\mu_m \mid m \geq 0)$ is a family with $\mu_m : \Sigma^{(m)} \to \mathcal{L}(V^m, V)$, the set of *multilinear* mappings from V^m to V
- final behaviour: $\gamma : V \to K$ is a linear form.

The Σ-algebra associated with (V, μ, γ) is $\mathcal{A}_V = (V, \overline{\Sigma}_\mu)$, where $\overline{\Sigma}_\mu = (\mu_m(\sigma) \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

$h_\mu : T_\Sigma \to \mathcal{A}_V$ is the unique Σ-homomorphism.

The tree series *recognized by* (V, μ, γ) is $\varphi \in K\langle\langle T_\Sigma \rangle\rangle$ such that $(\varphi, s) = \gamma(h_\mu(s))$ for every $s \in T_\Sigma$.

Tree series recognizable by multilinear mappings

1) (Example 4.1 of [BR82])The tree series size_δ is recognizable by multilinear mappings over the $\mathbb{Q}$-vector spece $V = (\mathbb{Q}^2, +, 0_2)$ with $0_2 = (0, 0)$.

Let (V, μ, γ) defined as follows:

For every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, $e_{i_1}, \ldots, e_{i_l} \in \{e_1 = (1, 0), e_2 = (0, 1)\}$, we define

$$\mu_m(\sigma)(e_{i_1}, \ldots, e_{i_m}) =$$
$$\begin{cases} e_1 + e_2 & \text{if } \sigma = \delta \text{ and } i_1 = \ldots = i_l = 1 \\ e_1 & \text{if } \sigma \neq \delta \text{ and } i_1 = \ldots = i_m = 1 \\ e_2 & \text{if there is exactly one } 1 \leq j \leq m \text{ with } i_j = 2 \\ 0_2 & \text{otherwise.} \end{cases}$$

$\gamma(e_1) = 0$, $\gamma(e_2) = 1$

For every $s \in T_\Sigma$, we have $h_\mu(s) = e_1 + \mathrm{size}_\delta(s) e_2$.

Tree series recognizable by multilinear mappings

2) (Example 9.2 of [BR82]) The tree series height is not recognizable by multilinear mappings over any $\mathbb{Q}$-vector space.

We denote the class of tree series recognizable by multilinear mappings over some K-vector space by $\mathrm{ML}(K)$.

Theorem. Every tree language which is recognizable by a deterministic tree automaton $M = (Q, \Sigma, F, \delta)$ is also recognizable by multilinear mappings over the $\mathbb{Z}_2$-vector space $\mathbb{Z}_2^Q$.

Proof. Let $Q = \{1, \ldots, n\}$, we define $(\mathbb{Z}_2^Q, \mu, \gamma)$ with

$$\mu_m(\sigma)(e_{i_1}, ..., e_{i_m}) = e_l \text{ iff } l = \delta_\sigma(i_1, ..., i_m),$$

$$\gamma(e_i) = 1 \text{ iff } i \in F.$$

K-Σ-tree automata [Boz99]

Preparation:

$(K, \oplus, \odot, 0, 1)$ a commutative semiring, $Q = \{q_1, \ldots, q_\kappa\}$ a finite set.

$(K^Q, +, 0_Q)$ is a K-semimodule via $\cdot : K \times K^Q \to K^Q$, with
$(k \cdot v)_q = k \odot v_q$

For $m \geq 0$ and $\nu : Q^m \to K^Q$, a *multilinear extension of* ν is a mapping
$\overline{\nu} : \underbrace{K^Q \times \ldots \times K^Q}_{m} \to K^Q$ such that

- $\overline{\nu}$ is multilinear

- $\overline{\nu}(1_{p_1}, \ldots, 1_{p_m}) = \nu(p_1, \ldots, p_m)$.

It is unique and

$$\overline{\nu}(v_1, \ldots, v_m)_q = \bigoplus_{p_1, \ldots, p_m \in Q} \left(\bigodot_{1 \leq i \leq m} (v_i)_{p_i} \right) \odot \nu(p_1, \ldots, p_m)_q.$$

K-Σ-tree automata [Boz99]

A system $M = (Q, \mu, f)$, where

- Q a finite set,
- $\mu = (\mu_m(\sigma) \mid m \geq 0, \sigma \in \Sigma^{(m)})$ is a family of transition functions $\mu_m(\sigma) : Q^m \to K^Q$, and
- $f : Q \to K$ is a final weight function.

For $m \geq 0$ and $\sigma \in \Sigma^{(m)}$, let $\overline{\mu_m(\sigma)} : (K^Q)^m \to K^Q$ be the multilinear extension of $\mu_m(\sigma)$.

The Σ-*algebra associated with* M is $\mathcal{A}_M = (K^Q, \overline{\Sigma}_\mu)$ where $\overline{\Sigma}_\mu = (\overline{\mu_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

The unique Σ-homomorphism from T_Σ to $\mathcal{A}_M$ is $h_\mu : T_\Sigma \to K^Q$.

The tree series recognized by M is $\varphi_M \in K\langle\langle T_\Sigma \rangle\rangle$ such that, for $s \in T_\Sigma$,

$$(\varphi_M, s) = \bigoplus_{q \in Q} h_\mu(s)_q \odot f(q).$$

K-Σ-tree automata [Boz99]

Example: the tree series height is recognizable by an $\underline{\mathrm{Arct}}$-$\Sigma$-tree automaton $M = (Q, \mu, f)$ where $\underline{\mathrm{Arct}} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ and

- $Q = \{p_1, p_2\}$,
- $f(p_1) = 0$ and $f(p_2) = -\infty$,
- μ is defined in the following way:
  - $\mu_0(\alpha)()_{p_1} = 0$,
  - $\mu_0(\alpha)()_{p_2} = 0$,
  - $\mu_2(\sigma)(p_1, p_2)_{p_1} = 1$,
  - $\mu_2(\sigma)(p_2, p_1)_{p_1} = 1$,
  - $\mu_2(\sigma)(p_2, p_2)_{p_2} = 0$,
  - $\mu_2(\sigma)(p, q)_r = -\infty$ for every other $p, q, r \in Q$.

K-Σ-tree automata [Boz99]

We denote the class of tree series recognizable by a K-Σ-tree automaton for some Σ by $\mathrm{TA}(K)$.

Theorem. For every field K, we have $\mathrm{ML}(K) = \mathrm{TA}(K)$.

Proof. Let $(V, +, 0)$ be a vector space over the field $(K, \oplus, \odot, 0, 1)$ of dimension $\kappa < \infty$; also let (V, μ, γ) be a multilinear representation of T_Σ. Moreover, let $M = (Q, \nu, f)$ be a K-Σ-ta over K. We say that (V, μ, γ) and M are *related* if

- Q is a basis of V,
- for every $m \geq 0$, $\sigma \in \Sigma^{(m)}$, and $p, p_1, \ldots, p_m \in Q$, the equation $\nu_m(\sigma)(p_1, \ldots, p_m)_p = \mu_m(\sigma)(p_1, \ldots, p_m)_p$ holds, and
- for every $p \in Q$, the equation $f(p) = \gamma(p)$ holds.

Weighted tree automata over semirings [AB87]

A system $M = (Q, \Sigma, K, F, \delta)$ (over the semiring $(K, \oplus, \odot, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in K$

$\delta = (\delta_m : \Sigma^{(m)} \to K^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$\delta_m(\sigma) \;=\; \begin{array}{c} \\ \vdots \\ q_1 \ldots q_m \\ \vdots \end{array} \begin{array}{c} \ldots \quad q \quad \ldots \\ \left[\begin{array}{ccc} & \vdots & \\ \ldots & a \in K & \ldots \\ & \vdots & \end{array} \right] \end{array} \in K^{Q^m \times Q}$$

Wta over semirings [AB87]

We define $\overline{\delta_m(\sigma)} : (K^Q)^m \to K^Q$, by

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1,\ldots,q_m \in Q} (v_1)_{q_1} \odot \ldots \odot (v_m)_{q_m} \odot \delta_m(\sigma)_{q_1 \ldots q_m, q}.$$

We associate $\mathcal{A}_M = (K^Q, \overline{\Sigma}_\delta)$, where $\overline{\Sigma}_\delta = (\overline{\delta_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

Let $h_\delta : T_\Sigma \to K^Q$ be the *unique* Σ*-homomorphism from* T_Σ *to* $\mathcal{A}_M$.

The tree language recognized by M is the tree series $\varphi_M : T_\Sigma \to K$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} h_\delta(s)_q \odot F_q.$$

Wta over semirings [AB87]

We denote the class of tree series recognizable by weighted tree automata over the semiring K by $\mathrm{WTA}(K)$.

Theorem. For every commutative semiring K, we have

$$\mathrm{TA}(K) = \mathrm{WTA}(K).$$

Corollary. For every field K, we have

$$\mathrm{ML}(K) = \mathrm{TA}(K) = \mathrm{WTA}(K).$$

Wta over semirings [AB87]

Determinization [BV03].

A system $M = (Q, \Sigma, K, F, \delta)$ (over the semiring $(K, \oplus, \odot, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in K$

$\delta = (\delta_m : \Sigma^{(m)} \to K^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$\delta_m(\sigma) \;=\; \begin{array}{c} \vdots \\ q_1 \ldots q_m \\ \vdots \end{array} \overset{\begin{array}{ccc} \ldots & q & \ldots \end{array}}{\left[\begin{array}{ccc} & \vdots & \\ \ldots & a \in K & \ldots \\ & \vdots & \end{array} \right]} \in K^{Q^m \times Q}$$

M is deterministic if, for every $q_1, \ldots, q_m \in Q$, there is at most one q with $\delta_m(\sigma)_{q_1 \ldots q_m, q} \neq 0$.

Wta over semirings [AB87]

Determinization.

In general wta over a semiring K and deterministic wta over K do not have the same recognizing power.

B. Borchardt and H. Vogler [BV03]:

- There is a wta over Trop which is not equivalent with any deterministic wta over Trop.
- They give a partial determinization algorithm, which converges in certain cases.

Finite tree automata over semirings with fixpoint semantics [Kui98, ÉK03]

The semiring $(K, \oplus, \odot, 0, 1)$ must be commutative,

- naturally ordered: $k \sqsubseteq k'$ iff $(\exists l \in K) k \oplus l = k'$ is a partial order,
- complete: infinite sum exists, and
- continuous: naturally ordered, complete and, for every ω-chain $k_1 \sqsubseteq k_2 \sqsubseteq \ldots$ in K and $k \in K$,

$$(\forall n \geq 1) \bigoplus_{i=1}^{n} k_i \sqsubseteq k \text{ implies that } \bigoplus_{i=1}^{\infty} k_i \sqsubseteq k.$$

Then K, $K\langle\langle T_\Sigma \rangle\rangle$, and $K\langle\langle T_\Sigma \rangle\rangle^n$ become a complete poset with respect to the (extension) of $\sqsubseteq$.

Finite tree automata over semirings with fixpoint semantics [Kui98, ÉK03]

A *finite tree automaton (over K and Σ)* is a tuple $M = (Q, \mathcal{M}, S, P)$ where

- Q is a finite set (of *states*),
- $\mathcal{M} = (\mathcal{M}_m \mid m \geq 0)$ is a family of *transition matrices* $\mathcal{M}_m$ such that $\mathcal{M}_m \in (K\langle T_\Sigma(Y_m)\rangle)^{Q \times Q^m}$ and for almost every m it holds that every entry of $\mathcal{M}_m$ is $\tilde{0}$,
- $S \in (K\langle T_\Sigma(Y_1)\rangle)^{1 \times Q}$ is the *initial state vector*, and
- $P \in (K\langle T_\Sigma\rangle)^{Q \times 1}$ is the *final state vector*.

Finite tree automata over semirings with fixpoint semantics [Kui98, ÉK03]

Such a system induces a *continuous* mapping

$$\Phi : K\langle\langle T_\Sigma \rangle\rangle^{Q\times 1} \to K\langle\langle T_\Sigma \rangle\rangle^{Q\times 1},$$

whose least fixpoint is fix Φ.

The *tree series recognized by* M is

$$\varphi_M = \bigoplus_{q\in Q} \big(S_q \leftarrow_{OI} (\mathrm{fix}\Phi)_q\big),$$

and we denote the class of tree series recognizable by finite tree automata over the semiring K with fixpoint semantics by $\mathrm{FTA}(K)$.

Theorem. For every commutative and continuous semiring K, we have

$$\mathrm{WTA}(K) = \mathrm{FTA}(K).$$

Polynomially weighted tree automata over semirings [Sei92]

A system $M = (Q, \Sigma, K, F, \delta)$ (over the semiring $(K, \oplus, \odot, 0, 1)$)

$F = (F_q \mid q \in Q)$ with $F_q \in P_1(K)$

$\delta = (\delta_m : \Sigma^{(m)} \to P_m(K)^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$\delta_m(\sigma) \;=\; \begin{array}{c} \\ \vdots \\ q_1 \ldots q_m \\ \vdots \end{array} \begin{array}{c} \ldots \quad q \quad \ldots \\ \left[\begin{array}{ccc} & \vdots & \\ \ldots & f \in P_m(K) & \ldots \\ & \vdots & \end{array} \right] \end{array} \in P_m(K)^{Q^m \times Q}$$

Polynomially weighted tree automata over semirings [Sei92]

We define $\overline{\delta_m(\sigma)} : (K^Q)^m \to K^Q$, by

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1,\ldots,q_m \in Q} \delta_m(\sigma)_{q_1 \ldots q_m, q}((v_1)_{q_1}, \ldots, (v_m)_{q_m}).$$

We associate $\mathcal{A}_M = (K^Q, \overline{\Sigma}_\delta)$, where $\overline{\Sigma}_\delta = (\overline{\delta_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

Let $h_\delta : T_\Sigma \to K^Q$ be the *unique* Σ*-homomorphism from* T_Σ *to* $\mathcal{A}_M$.

The tree language recognized by M is the tree series $\varphi_M : T_\Sigma \to K$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} F_q(h_\delta(s)_q).$$

Polynomially weighted tree automata over semirings [Sei92]

We denote the class of tree series recognizable by polynomially weighted tree automata over the semiring K by $\mathrm{PWTA}(K)$.

Theorem. For every semiring K, we have

$$\mathrm{WTA}(K) \subseteq \mathrm{PWTA}(K).$$

Theorem. $\mathrm{PWTA}(\mathbb{N}) - \mathrm{WTA}(\mathbb{N}) \neq \emptyset$.

Wta over M-monoids [Mal05, FMV06]

A *multioperator monoid* (for short: M-monoid) $(K, \oplus, 0, \Omega)$ consists of

- a commutative monoid $(K, \oplus, 0)$ and
- an Ω-algebra (K, Ω).

A multioperator monoid is *distributive* if

$$\omega_K(k_1, \ldots, k_{i-1}, \bigoplus_{j=1}^{n} a_j, k_{i+1}, \ldots, k_m) = \bigoplus_{j=1}^{n} \omega_K(k_1, \ldots, k_{i-1}, a_j, k_{i+1}, \ldots, k_m) \qquad \text{(d-}\Omega\text{)}$$

holds for every $m \geq 0$, $\omega \in \Omega^{(m)}$, $k_1, \ldots, k_m \in K$, $1 \leq i \leq m$, and $a_1, \ldots, a_n \in K$. (This implies $\omega_K(\ldots, 0, \ldots,) = 0$).

Wta over M-monoids [Mal05, FMV06]

A system $M = (Q, \Sigma, \underline{A}, F, \delta)$ (over the M-monoid $\underline{A} = (K, \oplus, \odot, \Omega)$)

$F = (F_q \mid q \in Q)$ with $F_q \in \Omega^{(1)}$

$\delta = (\delta_m : \Sigma^{(m)} \to (\Omega^{(m)})^{Q^m \times Q} \mid m \geq 0)$ of mappings.

$$\delta_m(\sigma) \;=\; \begin{array}{c} \\ \vdots \\ q_1 \ldots q_m \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \ldots & q & \ldots \end{array} \\ \left[\begin{array}{ccc} & \vdots & \\ \ldots & \omega \in \Omega^{(m)} & \ldots \\ & \vdots & \end{array} \right] \end{array} \in$$

$(\Omega^{(m)})^{Q^m \times Q}$

Wta over M-monoids [Mal05, FMV06]

We define $\overline{\delta_m(\sigma)} : (K^Q)^m \to K^Q$, by

$$\overline{\delta_m(\sigma)}(v_1, \ldots, v_m)_q = \bigoplus_{q_1, \ldots, q_m \in Q} \delta_m(\sigma)_{q_1 \ldots q_m, q}((v_1)_{q_1}, \ldots, (v_m)_{q_m}).$$

We associate $\mathcal{A}_M = (K^Q, \overline{\Sigma}_\delta)$, where $\overline{\Sigma}_\delta = (\overline{\delta_m(\sigma)} \mid m \geq 0, \sigma \in \Sigma^{(m)})$.

Let $h_\delta : T_\Sigma \to K^Q$ be the *unique* Σ*-homomorphism from* T_Σ *to* $\mathcal{A}_M$.

The tree language recognized by M is the tree series $\varphi_M : T_\Sigma \to K$ defined for every $s \in T_\Sigma$ by

$$(\varphi_M, s) = \bigoplus_{q \in Q} F_q(h_\delta(s)_q).$$

Wta over M-monoids [Mal05, FMV06]

We denote the class of tree series recognizable by weighted tree automata over the M-monoid $\underline{A}$ by $\mathrm{MWTA}(\underline{A})$.

Theorem. For every semiring K, we have $\mathrm{PWTA}(K) = \mathrm{MWTA}(\underline{\mathrm{Pol}(K)})$.

Theorem. $\mathrm{MWTA}(\underline{\mathbb{N}_{exp}}) - \mathrm{PWTA}(\mathbb{N}) \neq \emptyset$.

Theorem. (cf. Corollary 1 of [Mal05]) Let K be a distributive M-monoid and φ be a tree series which is recognizable by a <u>deterministic</u> wta over K. Then there is a semiring K' such that $K \subseteq K'$ and φ is recognizable by a wta over K'.

Wta over M-monoids [Mal05, FMV06]

A new result:

Theorem. ([FMV06]). Let K be a distributive M-monoid and φ be a tree series over K. Then φ is rational iff φ is recognizable.

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