# Weighted Tree Automata II. –
# A Kleene theorem for wta over M-monoids

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# Multioperator monoid

A *multioperator monoid* (for short: M-monoid) $(A, \oplus, 0, \Omega)$ consists of

- a commutative monoid $(A, \oplus, 0)$ and
- an $\Omega$-algebra $(A, \Omega)$
- with $\mathrm{id}_A \in \Omega^{(1)}$ and $0^m \in \Omega^{(m)}$ for $m \geq 0$.

$A$ is *distributive* if

$$\omega_A(b_1, \ldots, b_{i-1}, \bigoplus_{j=1}^{n} a_j, b_{i+1}, \ldots, b_m) = \bigoplus_{j=1}^{n} \omega_A(b_1, \ldots, b_{i-1}, a_j, b_{i+1}, \ldots, b_m)$$

holds for every $m, n \geq 0$, $\omega \in \Omega^{(m)}$, $b_1, \ldots, b_m \in A$, $1 \leq i \leq m$, and $a_1, \ldots, a_n \in A$. In particular, $\omega_A(\ldots, 0, \ldots,) = 0$.

# Operations on $\mathrm{Ops}(A)$

$\mathrm{Ops}(A)$ ($\mathrm{Ops}^k(A)$) is the set of operations ($k$-ary operations) on $A$.

Let $(A, \oplus, 0, \Omega)$ be an M-monoid and $k \geq 0$.

- Let $\omega_1, \omega_2 \in \mathrm{Ops}^k(A)$. The *sum of* $\omega_1$ *and* $\omega_2$ is the $k$-ary operation $\omega_1 \oplus \omega_2$ that is defined, for every $\vec{a} \in A^k$, by $(\omega_1 \oplus \omega_2)(\vec{a}) = \omega_1(\vec{a}) \oplus \omega_2(\vec{a})$.
- Let $\omega \in \mathrm{Ops}^k(A)$ and $\omega_j \in \mathrm{Ops}^{l_j}(A)$ with $l_j \geq 0$ for every $1 \leq j \leq k$. The *composition of* $\omega$ *with* $(\omega_1, \ldots, \omega_k)$ is the $(l_1 + \cdots + l_k)$-ary operation $\omega(\omega_1, \ldots, \omega_k)$ that is defined by

$$\big(\omega(\omega_1, \ldots, \omega_k)\big)(\vec{a_1}, \ldots, \vec{a_k}) = \omega(\omega_1(\vec{a_1}), \ldots, \omega_k(\vec{a_k}))$$

  for every $\vec{a_j} \in A^{l_j}$ with $1 \leq j \leq k$.

$(\mathrm{Ops}^k(A), \oplus, \mathbf{0}^k)$ is a commutative monoid for every $k \geq 0$, for $k = 0$ is isomorphic to the monoid $(A, \oplus, \mathbf{0})$.

Sum is left- and right- distributive, and composition is associative.

# Uniform tree valuations

$|t|_Z$ is the number of occurrences of variables of $Z$ in $t$

$\mathrm{Uvals}(\Sigma, Z, A)$ is the class of mappings $S : T_\Sigma(Z) \to \mathrm{Ops}(A)$ such that the arity of $(S, t)$ is $|t|_Z$. Such mappings are called uniform tree valuations over $\Sigma, Z$ and $A$.

- Hence $\mathrm{Uvals}(\Sigma, \emptyset, A) = A\langle\langle T_\Sigma \rangle\rangle$.
- $(\widetilde{\mathbf{0}}, t) = 0^{|t|_Z}$ for every $t \in T_\Sigma(Z)$.
- The *sum of* $S_1, S_2 \in \mathrm{Uvals}(\Sigma, Z, A)$ is the uniform tree valuation $S_1 \oplus^{\mathrm{u}} S_2$ defined by $(S_1 \oplus^{\mathrm{u}} S_2, t) = (S_1, t) \oplus (S_2, t)$ for every $t \in T_\Sigma(Z)$.
- $(\mathrm{Uvals}(\Sigma, Z, A), \oplus^{\mathrm{u}}, \widetilde{\mathbf{0}})$ is a commutative monoid; for $Z = \emptyset$ it is nothing but $(A\langle\langle T_\Sigma \rangle\rangle, \oplus, \widetilde{\mathbf{0}})$.
- For $S \in \mathrm{Uvals}(\Sigma, Z, A)$ we write $S = \bigoplus^{\mathrm{u}}_{t \in T_\Sigma(Z)} (S, t).t$.

# Weighted tree automata (wta) over M-monoids

Syntax

A system $M = (Q, \Sigma, Z, A, F, \mu, \nu)$ (over $\Sigma, Z$ and $A$)

- $Q, \Sigma, Z$ as before,
- $(A, \oplus, 0, \Omega)$ is an M-monoid,
- $F : Q \to \Omega^{(1)}$ is the root weight,
- $\mu = (\mu_m \mid m \geq 0)$ is the family of transition mappings with $\mu_m : Q^m \times \Sigma^{(m)} \times Q \to \Omega^{(m)}$,
- $\nu : Z \times Q \to \Omega^{(1)}$, the variable assignment.

Such a wta recognizes a *uniform tree valuation*, i.e., a mapping $S_M : T_\Sigma(Z) \to \mathrm{Ops}(A)$ in $\mathrm{Uvals}(\Sigma, Z, A)$.

In case $Z = \emptyset$ it recognizes a tree series in $A\langle\langle T_\Sigma \rangle\rangle$.

# Wta over M-monoids

Semantics

$M = (Q, \Sigma, Z, A, F, \mu, \nu)$ a wta over the M-monoid $A$ and $t \in T_\Sigma(Z)$

- a run of $M$ on $t$ is a mapping $r : \mathrm{pos}(t) \to Q$

- the set of runs of $M$ on $t$ is $R_M(t)$

- for $w \in \mathrm{pos}(t)$, the weight $\mathrm{wt}(t, r, w)$ of $w$ in $t$ under $r$

- if $t(w) = z$ for some $z \in Z$, then $\mathrm{wt}(t, r, w) = \nu(z, r(w))$
- otherwise (if $t(w) = \sigma$ for some $\sigma \in \Sigma^{(k)}, k \geq 0$) $\mathrm{wt}(t, r, w) = \mu_k(r(w1), \ldots, r(wk), t(w), r(w))(\mathrm{wt}(t, r, w1), \ldots, \mathrm{wt}(t, r, wk))$
- the weight of $r$ is $\mathrm{wt}(t, r) = \mathrm{wt}(t, r, \varepsilon)$.

The uniform tree valuation $S_M : T_\Sigma(Z) \to A$ recognized by $M$ is defined by

$$S_M(t) = \bigoplus_{r \in R_M(t)} F(r(\varepsilon))(\mathrm{wt}(t, r)).$$

# An example of a wta over M-monoids

The tree series $\mathrm{height} : T_\Sigma \to \mathbb{N}$ can be recognized by

$$M = (Q, \Sigma, A, F, \mu),$$

where

- $Q = \{q\}$,
- $A = (\mathbb{N}, -, -, \Omega)$ with $\{1 + \max\{n_1, \ldots, n_k\} \mid k \geq 0\} \subseteq \Omega$,
- $F(q) = \mathrm{id}_\mathbb{N}$, and
- $\mu_0(\alpha, q) = 0$ and for every $k \geq 1$ and $\sigma \in \Sigma^{(k)}$, let $\mu_k(q \ldots q, \sigma, q) = 1 + \max\{n_1, \ldots, n_k\}$.

Then $S_M = \mathrm{height}$.

# Rational operations on $\mathrm{Uvals}(\Sigma, Z, A)$

1. The *sum* $\oplus^{\mathrm{u}}$ : $(S_1 \oplus^{\mathrm{u}} S_2, t) = (S_1, t) \oplus (S_2, t)$.

2. The *top-concatenation*: for $k \geq 0$, $\sigma \in \Sigma^{(k)}, \omega \in \Omega^{(k)}$, and $S_1, \ldots, S_k \in \mathrm{Uvals}(\Sigma, Z, A)$, we define

$$\mathrm{top}_{\sigma,\omega}(S_1, \ldots, S_k) = \bigoplus^{\mathrm{u}}_{t_1,\ldots,t_k \in T_\Sigma(Z)} \omega((S_1, t_1), \ldots, (S_k, t_k)).\sigma(t_1, \ldots, t_k).$$

3. The $z$-*concatenation*: for every $z \in Z$ and $S, S' \in \mathrm{Uvals}(\Sigma, Z, A)$, we define

$$S \cdot_z S' = \bigoplus^{\mathrm{u}}_{\substack{s \in T_\Sigma(Z),\, l = |s|_z \\ t_1,\ldots,t_l \in T_\Sigma(Z)}} \Big((S, s) \circ_{s,z} ((S', t_1), \ldots, (S', t_l))\Big).s[z \leftarrow (t_1, \ldots, t_l)] \ .$$

# Rational operations on $\mathrm{Uvals}(\Sigma, Z, A)$

4. The $z$-KLEENE-*star*: for every $z \in Z$ and $S \in \mathrm{Uvals}(\Sigma, Z, A)$ we define:
   (i) $S_z^0 = \widetilde{\mathbf{0}}$; and
   (ii) $S_z^{n+1} = (S \cdot_z S_z^n) \oplus^{\mathrm{u}} \mathrm{id}_A.z$.

   Then, the $z$-KLEENE star $S_z^*$ of $S$ is defined as follows:

   If $S$ is $z$-proper, i.e., $(S, z) = \mathbf{0}$, then

$$(S_z^*, t) = (S_z^{\mathrm{height}(t)+1}, t)$$

   for every $t \in T_\Sigma(Z)$, otherwise $S_z^* = \widetilde{\mathbf{0}}$.

# Rational expressions (over $\Sigma, Z$ and $A$)

$\mathrm{RatExp}(\Sigma, Z, A)$ (over $\Sigma$, $Z$, and $A$) is the smallest set $R$ satisfying Conditions (i)–(v). For every ratexp $\eta \in \mathrm{RatExp}(\Sigma, Z, A)$ we define its semantics $[\![\eta]\!] \in \mathrm{Uvals}(\Sigma, Z, A)$ simultaneously.

(i) For every $z \in Z$ and $\omega \in \Omega^{(1)}$ we have $\omega.z \in R$ and $[\![\omega.z]\!] = \omega.z$.

(ii) For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, $\omega \in \Omega^{(k)}$, and rational expressions $\eta_1, \ldots, \eta_k \in R$ we have $\mathrm{top}_{\sigma,\omega}(\eta_1, \ldots, \eta_k) \in R$ and $[\![\mathrm{top}_{\sigma,\omega}(\eta_1, \ldots, \eta_k)]\!] = \mathrm{top}_{\sigma,\omega}([\![\eta_1]\!], \ldots, [\![\eta_k]\!])$.

(iii) For every $\eta_1, \eta_2 \in R$ we have $\eta_1 + \eta_2 \in R$ and $[\![\eta_1 + \eta_2]\!] = [\![\eta_1]\!] \oplus^{\mathrm{u}} [\![\eta_2]\!]$.

(iv) For every $\eta_1, \eta_2 \in R$ and $z \in Z$ we have $\eta_1 \cdot_z \eta_2 \in R$ and $[\![\eta_1 \cdot_z \eta_2]\!] = [\![\eta_1]\!] \cdot_z [\![\eta_2]\!]$.

(v) For every $\eta \in R$ and $z \in Z$ we have $\eta_z^* \in R$ and $[\![\eta_z^*]\!] = [\![\eta]\!]_z^*$.

# Rational tree valuations (over $\Sigma, Z$ and $A$)

We call $S \in \mathrm{Uvals}(\Sigma, Z, A)$ *rational*, if there exists a rational expression $\eta \in \mathrm{RatExp}(\Sigma, Z, A)$ such that $[\![\eta]\!] = S$.

$\mathrm{Rat}(\Sigma, Z, A)$ is the class of rational uniform tree valuations over $\Sigma, Z$ and $A$.

Then $\mathrm{Rat}(\Sigma, Z, A)$ is the smallest class of uniform tree valuations which contains the uniform tree valuation $\omega.z$ for every $z \in Z$ and $\omega \in \Omega^{(1)}$ and is closed under the rational operations.

# Kleene theorem for wta over M-monoids

a) Recognizable $\Rightarrow$ rational:

Theorem. If $A$ is distributive, then for every wta $M = (Q, \Sigma, Z, A, F, \mu, \nu)$ there exists a rational expression $\eta \in \mathrm{RatExp}(\Sigma, Z \cup Q, A)$ such that $S_M = [\![\eta]\!]|_{T_\Sigma(Z)}$.

Hence we have $\mathrm{Rec}(\Sigma, Z, A) \subseteq \mathrm{Rat}(\Sigma, \mathrm{fin}, A)|_{T_\Sigma(Z)}$, where

$$\mathrm{Rat}(\Sigma, \mathrm{fin}, A) = \bigcup_{Z \text{ finite set}} \mathrm{Rat}(\Sigma, Z, A).$$

# Kleene theorem for wta over M-monoids

The M-monoid $(A, \oplus, 0, \Omega)$ is

- *sum closed*, if $\omega_1 \oplus \omega_2 \in \Omega^{(k)}$ for every $k \geq 0$ and $\omega_1, \omega_2 \in \Omega^{(k)}$.
- $(1, \star)$-*composition closed*, if $\omega(\omega') \in \Omega^{(k)}$ for every $k \geq 0$, $\omega \in \Omega^{(1)}$, and $\omega' \in \Omega^{(k)}$.
- $(\star, 1)$-*composition closed*, if $\omega(\omega_1, \ldots, \omega_k) \in \Omega^{(k)}$ for every $k \geq 0$, $\omega \in \Omega^{(k)}$, and $\omega_1, \ldots, \omega_k \in \Omega^{(1)}$.

b) Rational $\Rightarrow$ recognizable:

Theorem. Let $A$ be a distributive, $(1, \star)$-composition closed and sum closed. Then $\mathrm{Rec}(\Sigma, Z, A)$ contains the uniform tree valuation $\omega.z$ for every $z \in Z$ and $\omega \in \Omega^{(1)}$, and it is closed under the rational operations.
Hence, $\mathrm{Rat}(\Sigma, Z, A) \subseteq \mathrm{Rec}(\Sigma, Z, A)$.

# Kleene theorem for wta over M-monoids

In case $Z = \emptyset$:

Theorem. For every $(1,\star)$-composition closed and sum closed DM-monoid $A$, we have $\mathrm{Rec}(\Sigma, \emptyset, A) = \mathrm{Rat}(\Sigma, \mathrm{fin}, A)|_{T_\Sigma}$.

Proof. We have

$$\mathrm{Rec}(\Sigma, \emptyset, \underline{A}) \subseteq \mathrm{Rat}(\Sigma, \mathrm{fin}, A)|_{T_\Sigma} \subseteq \mathrm{Rec}(\Sigma, \mathrm{fin}, A)|_{T_\Sigma} \subseteq \mathrm{Rec}(\Sigma, \emptyset, A)$$

where the last inclusion can be seen as follows. Let $S \in \mathrm{Rec}(\Sigma, \mathrm{fin}, A)|_{T_\Sigma}$. Thus, there exist a wta $M = (Q, \Sigma, Z, A, F, \mu, \nu)$ such that $S = S_M|_{T_\Sigma}$. It is easy to see that for the wta $N = (Q, \Sigma, \emptyset, A, F, \mu, \emptyset)$ we have that $S_N = S_M|_{T_\Sigma}$. Thus $S \in \mathrm{Rec}(\Sigma, \emptyset, A)$.

# Wta over (arbitrary) semirings

$M = (Q, \Sigma, Z, K, F, \delta, \nu)$ a wta, $K$ is a semiring, $t \in T_\Sigma(Z)$

- a run of $M$ on $t$ is a mapping $r : \mathrm{pos}(t) \to Q$

- the set of runs of $M$ on $t$ is $R_M(t)$

- for $w \in \mathrm{pos}(t)$, the weight $\mathrm{wt}(t, r, w)$ of $w$ in $t$ under $r$

- if $t(w) = z$ for some $z \in Z$, then $\mathrm{wt}(t, r, w) = \nu(z, r(w))$
- otherwise (if $t(w) = \sigma$ for some $\sigma \in \Sigma^{(k)}, k \geq 0$)
  $\mathrm{wt}(t, r, w) = \delta_k(r(w1), \ldots, r(wk), t(w), r(w))$
- the weight of $r$ is $\mathrm{wt}(t, r) = \prod_{w \in pos(t)} \mathrm{wt}(t, r, w)$, where the order of the product is the postorder tree walk.

The tree series $S_M : T_\Sigma(Z) \to K$ recognized by $M$ is

$$S_M(t) = \sum_{r \in R_M(t)} \mathrm{wt}(t, r) \cdot F(r(\varepsilon)).$$

The class of recognizable tree series by such wta: $\mathrm{Rec}_{\mathrm{sr}}(\Sigma, Z, K)$.

# Semiring M-monoids

An arbitrary semiring $(K, \oplus, \odot, 0, 1)$ can be simulated by an appropriate M-monoid:

for every $a \in K$, let $\mathrm{mul}_a^{(k)} \colon K^k \to K$ be the mapping defined as follows: for every $a_1, \ldots, a_k \in K$ we have $\mathrm{mul}_a^{(k)}(a_1, \ldots, a_k) = a_1 \odot \cdots \odot a_k \odot a$.

Moreover, let $\underline{D}(K) = (K, \oplus, \mathbf{0}, \Omega)$, where $\Omega^{(k)} = \{\mathrm{mul}_a^{(k)} \mid a \in K\}$.

Then $\underline{D}(K)$ is a distributive, sum closed, and $(1, \star)$-composition closed M-monoid. ($\mathrm{id}_K = \mathrm{mul}_1^{(1)}$ and $0^k = \mathrm{mul}_0^{(k)}$.)

Theorem. $\mathrm{Rec}_{\mathrm{sr}}(\Sigma, Z, K) = \mathrm{Rec}(\Sigma, Z, \underline{D}(K))$.

# A Kleene theorem for wta over arbitrary semirings

Theorem. $\mathrm{Rec}_{\mathrm{sr}}(\Sigma, K) = \mathrm{Rat}(\Sigma, \mathrm{fin}, \underline{D}(K))|_{T_\Sigma}$ for every semiring $K$.

Proof.

$$\mathrm{Rec}_{\mathrm{sr}}(\Sigma, K) = \mathrm{Rec}(\Sigma, \emptyset, \underline{D}(K)) = \mathrm{Rat}(\Sigma, \mathrm{fin}, \underline{D}(K))|_{T_\Sigma}$$

# Rational tree series over a semiring $K$

The set of *rational tree series expressions over* $\Sigma$*,* $Z$ *and* $K$, denoted by $\mathrm{RatExp}(\Sigma, Z, K)$, is the smallest set $R$ which satisfies Conditions (1)-(6). For every $\eta \in \mathrm{RatExp}(\Sigma, Z, K)$ we define $[\![\eta]\!]_{\mathrm{sr}} \in K\langle\!\langle T_\Sigma(Z)\rangle\!\rangle$ simultaneously.

1. For every $z \in Z$, the expression $z \in R$, and $[\![z]\!]_{\mathrm{sr}} = 1.z$.
2. For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$, and $\eta_1, \ldots, \eta_k \in R$, the expression $\sigma(\eta_1, \ldots, \eta_k) \in R$ and $[\![\sigma(\eta_1, \ldots, \eta_k)]\!]_{\mathrm{sr}} = \mathrm{top}_\sigma([\![\eta_1]\!]_{\mathrm{sr}}, \ldots, [\![\eta_k]\!]_{\mathrm{sr}})$.
3. For every $\eta \in R$ and $a \in K$, the expression $(a\eta) \in R$ and $[\![(a\eta)]\!]_{\mathrm{sr}} = a[\![\eta]\!]_{\mathrm{sr}}$.
4. For every $\eta_1, \eta_2 \in R$, the expression $(\eta_1 + \eta_2) \in R$ and $[\![(\eta_1 + \eta_2)]\!]_{\mathrm{sr}} = [\![\eta_1]\!]_{\mathrm{sr}} + [\![\eta_2]\!]_{\mathrm{sr}}$.
5. For every $\eta_1, \eta_2 \in R$ and $z \in Z$, the expression $(\eta_1 \circ_z \eta_2) \in R$ and $[\![(\eta_1 \circ_z \eta_2)]\!]_{\mathrm{sr}} = [\![\eta_1]\!]_{\mathrm{sr}} \circ_z [\![\eta_2]\!]_{\mathrm{sr}}$.
6. For every $\eta \in R$ and $z \in Z$, the expression $(\eta_z^*) \in R$ and $[\![(\eta_z^*)]\!]_{\mathrm{sr}} = [\![\eta]\!]_{\mathrm{sr},z}^*$.

The class of rational tree series: $\mathrm{Rat}_{\mathrm{sr}}(\Sigma, Z, K)$.

# A Kleene theorem for wta over commutative semirings

We can relate rational tree series over $\Sigma$, $Z$, and $K$ and rational uniform tree valuations over $\Sigma$, $Z$, and $\underline{D}(\underline{K})$.

For this, define:

$\mathrm{one} : \mathrm{Umaps}(\Sigma, Z, \underline{D}(\underline{K})) \to K\langle\langle T_{\Sigma(Z)}\rangle\rangle$ as follows.

For every $S \in \mathrm{Umaps}(\Sigma, Z, \underline{D}(\underline{K}))$ and $t \in T_{\Sigma\cup Z}$, let $(\mathrm{one}(S), t) = (S, t)(1, \ldots, 1)$, where the number of arguments $1$ is $|t|_Z$.

Note that $(\mathrm{one}(S), t) = (S, t)$ for every $t \in T_\Sigma$. We extend $\mathrm{one}$ to classes in the usual way.

Lemma. For every <u>commutative</u> semiring $K$, we have $\mathrm{Rat}_{\mathrm{sr}}(\Sigma, Z, K) = \mathrm{one}(\mathrm{Rat}(\Sigma, Z, \underline{D}(\underline{K})))$.

# A Kleene theorem for wta over commutative semirings

Corollary. For every commutative semiring $K$, we have that
$\mathrm{Rec}_{\mathrm{sr}}(\Sigma, K) = \mathrm{Rat}_{\mathrm{sr}}(\Sigma, \mathrm{fin}, K)|_{T_\Sigma}$.

Proof.

a) $\mathrm{Rat}_{\mathrm{sr}}(\Sigma, Z, K)|_{T_\Sigma} = \mathrm{one}(\mathrm{Rat}(\Sigma, Z, \underline{D}(\underline{K})))|_{T_\Sigma} = \mathrm{Rat}(\Sigma, Z, \underline{D}(\underline{K}))|_{T_\Sigma}$

Then

$\mathrm{Rat}_{\mathrm{sr}}(\Sigma, \mathrm{fin}, \underline{K})|_{T_\Sigma} = \mathrm{Rat}(\Sigma, \mathrm{fin}, \underline{D}(\underline{K}))|_{T_\Sigma}$

b) We already proved

$\mathrm{Rec}_{\mathrm{sr}}(\Sigma, K) = \mathrm{Rec}(\Sigma, \emptyset, \underline{D}(K)) = \mathrm{Rat}(\Sigma, \mathrm{fin}, \underline{D}(K))|_{T_\Sigma}$

# Kleene theorem for wta over commutative semirings

Lemma. For every commutative semiring $K$, we have
$\mathrm{Rat}_{\mathrm{sr}}(\Sigma, Z, K) = \mathrm{one}(\mathrm{Rat}(\Sigma, Z, \underline{D}(\underline{K})))$.

Proof. We redefine rational expressions over $\Sigma, Z$ and $\underline{D}(\underline{K})$:

$\mathrm{RatExp}'(\Sigma, Z, \underline{D}(\underline{K}))$ and $\mathrm{Rat}'(\Sigma, Z, \underline{D}(\underline{K}))$

(i) For every $z \in Z$ we have $z \in R$ and $[\![z]\!] = \mathrm{mul}_1^{(1)}.z$.

(ii) For every $k \geq 0$, $\sigma \in \Sigma^{(k)}$ and rational expressions $\eta_1, \ldots, \eta_k \in R$ we have $\sigma(\eta_1, \ldots, \eta_k) \in R$ and $\sigma(\eta_1, \ldots, \eta_k)]\!] = \mathrm{top}_{\sigma, \mathrm{mul}_1^{(k)}}([\![\eta_1]\!], \ldots, [\![\eta_k]\!])$.

(iii) For every $\eta \in R$ and $a \in K$, the expression $(a\eta) \in R$ and $[\![(a\eta)]\!] = \mathrm{mul}_a^{(1)} \circ [\![\eta]\!]$.

(iv) For every $\eta_1, \eta_2 \in R$ we have $\eta_1 + \eta_2 \in R$ and $[\![\eta_1 + \eta_2]\!] = [\![\eta_1]\!] \oplus^{\mathrm{u}} [\![\eta_2]\!]$.

(v) For every $\eta_1, \eta_2 \in R$ and $z \in Z$ we have $\eta_1 \cdot_z \eta_2 \in R$ and $[\![\eta_1 \cdot_z \eta_2]\!] = [\![\eta_1]\!] \cdot_z [\![\eta_2]\!]$.

(vi) For every $\eta \in R$ and $z \in Z$ we have $\eta_z^* \in R$ and $[\![\eta_z^*]\!] = [\![\eta]\!]_z^*$.

# Kleene theorem for wta over commutative semirings

Then

$$\mathrm{Rat}'(\Sigma, Z, \underline{D}(\underline{K})) = \mathrm{Rat}(\Sigma, Z, \underline{D}(\underline{K})) \text{ and}$$
$$\mathrm{RatExp}'(\Sigma, Z, \underline{D}(\underline{K})) = \mathrm{RatExp}(\Sigma, Z, K).$$

Thus we can prove by induction on $\eta$: for every $\eta \in \mathrm{RatExp}'(\Sigma, Z, \underline{D}(\underline{K}))$, $t \in T_\Sigma(Z)$, and $a_1, \ldots, a_n \in K$, we have that

$$(\llbracket \eta \rrbracket, t)(a_1, \ldots, a_n) = (\llbracket \eta \rrbracket_{\mathrm{sr}}, t) \odot a_1 \odot \ldots \odot a_n.$$

This implies that for every $\eta \in \mathrm{RatExp}'(\Sigma, Z, \underline{D}(\underline{K}))$, we have $\llbracket \eta \rrbracket_{\mathrm{sr}} = \mathrm{one}(\llbracket \eta \rrbracket)$, where $\llbracket \eta \rrbracket_{\mathrm{sr}}$ denotes the semiring semantics of $\eta$.

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